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The decay of transients in a contained stratified fluid

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Stably stratified viscous fluid in a container with vertical walls is initially at rest with tilted density surfaces, following a rotation of the container from its orientation when being filled. The initial state so generated is not in equilibrium, and the resultant motion will decay under the action of viscosity and of the diffusion of the salt producing the stratification. The timescales for the succession of stages by which equilibrium is attained are identified, and are found to depend on the strengths of the two diffusive processes, separately and interactively.

1. Introduction

If a stratified fluid is in a container with vertical walls, any initial distortion of the surfaces of constant density away from the horizontal will generate internal waves. The motion will continue until diffusive effects bring the fluid to a state of equilibrium in which it will be at rest with horizontal density surfaces. Of the two diffusive processes at work, viscosity and density diffusion, the former is usually the more active; the ratio of the respective diffusivities is about 560 for salt water, for example. The main purpose of this paper is to determine the timescales for the achievement of equilibrium in a geometrically simple configuration. The most important conclusion is that the motion decays in a timescale inversely proportional to the viscosity, as expected, but the timescale for the density surfaces to become horizontal also depends on a parameter involving both diffusivities. Either of these scales may be the larger, depending on the values of the diffusivities and other parameters.

The corresponding problem when the walls of the container are not vertical is quite different. The state of the fluid in which the density surfaces are horizontal is then not an equilibrium state. The boundary condition on the density at the wall induces density gradients in the fluid; the resulting buoyancy layers were discussed by Phillips (1970) and Wunsch (1970). In this paper only the vertical-wall case is considered.

Before examining the later development of an initial state in which the density surfaces are inclined to the horizontal, it is necessary to see if it can be produced in a realistic manner. One possibility is to fill the container with its walls non-vertical and then rotate it to the desired orientation. This motion of the container will, in general, alter the disposition of the fluid relative to the container, and may result in a state in which the density surfaces are inclined to the horizontal and the walls are vertical. A translation of the container with an acceleration slow enough to avoid the excitation of internal waves will force the fluid to move rigidly with the container, and no tilting of the density surfaces will result. In a rotation, however, the fluid cannot move rigidly with the container, since the movement of the boundary can only impart an irrotational motion to the fluid, assuming that the motion is sufficiently rapid for viscous action to be ineffective. Such a rotation occurs in some laboratory experimentation with stratified fluids, and it then becomes necessary to estimate the extent of the resultant distortion of the density surfaces and the timescale for the re-establishment of equilibrium. The purpose of this paper is to provide a theoretical prediction of these two quantities.

An example of an experiment in which a container was rotated is described by Thorpe (1968). His apparatus consisted of a long rectangular tank placed with the long walls horizontal and filled with stratified fluid. The tank was rotated to an inclined position, and the density surfaces did not move relative to the tank. The inclined density surfaces then induced a shear flow along the tank as the lighter fluid displaced the heavier fluid above it. In contrast, the initial state after the rotation considered here is when the density surfaces are neither horizontal nor vertical and the walls of the tank are vertical. The density imbalance produces internal waves without a bulk motion of the fluid, and these waves eventually decay.

There are two parts to the investigation described here. In the first, the non-diffusive effect of a change in orientation of a rectangular container on the fluid inside it is evaluated. Since diffusive effects are ignored here, it is essential that both the rotation and the filling of the container take place over a time short compared with the time for a diffusive boundary layer to appear. In its pre-rotation position, the walls of the container will not be vertical, even for the special case of plane walls, and buoyancy layers of the type considered by Phillips (1970) and Wunsch (1970) will be produced, albeit quite slowly. The second, and more important, part of the paper concerns the subsequent behaviour of the fluid as internal waves are generated and decay under the action of the diffusion of both vorticity and density. The large difference in the strengths of these two diffusive processes leads to a variety of timescales for the stages through which the motion passes on its way to equilibrium and the identification of these timescales is the goal of the work described in §§3 and 4. The paper ends with some further remarks and conclusions.

2. Tilting of a rectangular container

A long and wide rectangular tank is initially placed as shown in figure 1, with its walls at an angle θ to the vertical. If (X, Y) are Cartesian axes fixed relative to the tank, the walls are in the planes $X = \pm a$ and the axis of rotation is normal to the (X, Y)-plane. Throughout, the motion is two-dimensional, with no variation in the direction of the axis of rotation. The angular velocity imparted to the tank is $\omega(t)$, so that the angle $\phi(t)$ through which the tank is rotated is given by

$$\phi = \int_0^t \omega(t_1)\,\mathrm{d}t_1. \tag{2.1}$$

If the rotation lasts for a total time T, the total angle turned through is $\Phi = \phi(T)$.

The initial density distribution of the continuously and linearly stratified fluid is given by

$$\rho_0 = \rho_c - \beta(y\cos\theta - x\sin\theta), \tag{2.2}$$

where ρ_c and β are positive constants. In the rotating frame fixed to the container, we can write the velocity components (u, v) in the (x, y)-directions in terms of a stream function ψ so that

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x}. \tag{2.3}$$

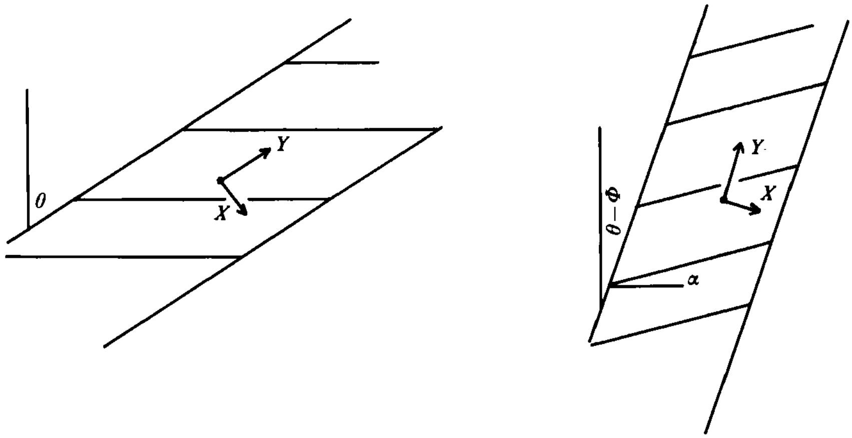

FIGURE 1. Sketch of the container and the surfaces of constant density before and after the container has been rotated through an angle Φ.

By making the rotation rate sufficiently large, we can ensure that the rotation is completed in a time less than the diffusive timescales identified in §3. Then the vorticity generated by the stratification and by viscosity are negligible, and it follows that

$$\nabla^2\psi - 2\omega = 0, \tag{2.4}$$

which simply states that the motion is irrotational in a frame fixed in space. When the rotation rate is large, the diffusion of density during the rotation can also be neglected and the density is advected with the fluid. Hence to determine the density distribution at time T, which is the objective in this section, it is necessary to find the displacement of an arbitrary fluid particle during the rotation.

The boundary condition on ψ is that $\psi = 0$ on the boundary of the container, and the solution of (2.4) under this condition is simply given by

$$\psi = \omega(x^2 - a^2), \tag{2.5}$$

with the velocity components given by

$$u = 0, \quad v = -2\omega x. \tag{2.6}$$

The position (x, y) at time t of the fluid particle that is at (x_T, y_T) at time T is given by

$$x = x_T, \quad y = y_T + 2x_T \int_t^T \omega(t_1)\, \mathrm{d}t_1, \tag{2.7}$$

and, if we use a suffix 0 to denote the position at $t = 0$, we see that

$$x_0 = x_T, \quad y_0 = y_T + 2\Phi x_T. \tag{2.8}$$

The initial density at an arbitrary point is given by (2.2), and so we find the density ρ_T at time T of the fluid particle that is then at (x_T, y_T) is given by

$$\rho_T = \rho_c - \beta\{(y_T + 2\Phi x_T)\cos\theta - x_T \sin\theta\}. \tag{2.9}$$

The isopycnals therefore remain planar throughout the rotation, and they are inclined at an angle Ψ to the x-axis after the container has turned through an angle Φ, where

$$\tan\Psi = \tan\theta - 2\Phi. \tag{2.10}$$

To determine the orientation of the isopycnals in space, we must add to the angle Ψ the angle through which the container has rotated and subtract the initial inclination of the x-axis to the horizontal. Thus the angle α through which the isopycnals have been tilted by the rotation of the container is given by

$$\alpha = \Phi - \theta + \tan^{-1}(\tan\theta - 2\Phi). \tag{2.11}$$

The experiments conducted by Thorpe (1968) correspond to the special case $\theta = \frac{1}{2}\pi$. Then $\alpha = \Phi$, and the isopycnals rotate with the container. This does not mean, of course, that the fluid is rotating with the container. The fluid moves parallel to the container walls, and there is no density variation in this direction. In recent experiments by Dr P. F. Linden and Dr J. E. Simpson (private communication) a tank with walls initially vertical was rotated through 90°. The prediction of (2.11) in this case is that

$$\alpha = \tfrac{1}{2}\pi - \tan^{-1}\pi \approx 17.6°, \tag{2.12}$$

which is confirmed by the observations made by Linden and Simpson.

The case of interest in §3 is when the container is rotated until the walls are vertical. Thus $\Phi = \theta$, and (2.11) gives the result

$$\alpha = \tan^{-1}(\tan\theta - 2\theta). \tag{2.13}$$

For values of $\theta < 66.78°$ the isopycnals are displaced in the sense opposite to the rotation, with a maximum deflection of 29.7° when $\theta = 45°$. For $\theta > 66.78°$ the deflection follows the sense of the rotation, and the inclination of the isopycnals increases rapidly; they become vertical when $\theta = 90°$.

It should be noted that the state of the fluid after the rotation is given by the density distribution determined in the above analysis and that there is no resultant motion. In reality, there will be a small residual velocity because of the weak action of viscosity at the boundary, where vortex sheets of opposite strengths have been generated by starting and stopping the rotation. With sufficiently rapid rotation these vortex sheets almost cancel, and the resultant motion can be ignored.

3. Decay of sloshing motion in a container with vertical walls

At the end of §2 we obtained an expression for the tilt of the density surfaces in a rectangular container when it was rotated until the walls became vertical. We now consider how this initial state develops with time, the container being held stationary. The tilted density surfaces initiate a sloshing motion of the fluid, which is damped by viscosity. If this is the only diffusive process acting, the fluid in contact with the wall retains its initial density, since it does not move relative to the wall. Hence large density gradients are induced close to the wall. Although the diffusion of the material producing the stratification is much weaker than the diffusion of vorticity by viscosity, these large density gradients indicate that density diffusion cannot be neglected. In this section we consider the effects of both diffusive processes, and, in particular, determine the timescales for the different stages by which an equilibrium state is attained.

The problem to be studied as a simple example for which the important processes can be identified is as follows. We suppose that the fluid is contained between the vertical walls $X = \pm a$, and that the initial density is given by

$$R = \rho_c - \rho_c \beta'(Y - X\tan\alpha)/a, \tag{3.1}$$

so that the density surfaces are inclined at an angle α to the horizontal, and that the fluid is initially at rest. The reference density ρ_c and the density stratification β' are constants. The dimensional spatial variables X (horizontal) and Y (vertical) and the time T' are made non-dimensional by writing

$$X = ax, \quad Y = ay, \quad T' = \left(\frac{\beta' g}{a}\right)^{-\frac{1}{2}} t. \tag{3.2}$$

The velocity components (U, V) in the (X, Y)-directions are written as

$$(U, V) = (\beta' g a)^{\frac{1}{2}} \tan\alpha \,(u, v), \tag{3.3}$$

and the pressure P and density R as

$$P = P_c - \rho_c g Y + \rho_c \beta' g a \tan\alpha \, p, \tag{3.4}$$

$$R = \rho_c - \rho_c \beta' (Y - a \tan\alpha \, \rho)/a, \tag{3.5}$$

so that p and ρ are the non-dimensional measures of the perturbation to the pressure and density fields associated with the basic vertical stratification. If we make the Oberbeck–Boussinesq approximation, assuming that variations of density in the Navier–Stokes equations are only important in the buoyancy term, and that the diffusion coefficients can be regarded as constants, the equations satisfied by the dependent variables u, v, p and ρ are

$$\frac{\partial u}{\partial t} + \tan\alpha\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \lambda\, \nabla^2 u, \tag{3.6}$$

$$\frac{\partial v}{\partial t} + \tan\alpha\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \lambda\, \nabla^2 v - \rho, \tag{3.7}$$

$$\frac{\partial \rho}{\partial t} + \tan\alpha\left(u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right) = \mu\, \nabla^2 \rho + v, \tag{3.8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{3.9}$$

where

$$\lambda = \frac{\nu}{(\beta' g a^3)^{\frac{1}{2}}}, \quad \mu = \frac{\kappa}{(\beta' g a^3)^{\frac{1}{2}}}. \tag{3.10}$$

The importance of the two diffusive processes is measured by the values of these two parameters. We shall only consider the case when these effects are both small, so that λ and μ are small compared with unity, and we shall also assume that the diffusion of density is much weaker than that of vorticity, so that $\lambda \gg \mu$. As mentioned previously, for salt water λ/μ, or ν/κ, is approximately 560. The boundary conditions which the solutions of (3.6)–(3.9) must satisfy are

$$u = v = 0, \quad \frac{\partial \rho}{\partial x} = 0 \quad \text{at } x = \pm 1, \tag{3.11}$$

and the initial conditions are that, at time $t = 0$,

$$u = v = 0, \quad \rho = x. \tag{3.12}$$

These boundary and initial conditions and the equations themselves show that a solution is possible in which the velocity is vertical and the velocity and density depend only on the time t and the horizontal coordinate x. Thus we can write

$$u = 0, \quad v = v(x, t), \quad \rho = \rho(x, t), \tag{3.13}$$

and (3.6)–(3.9) then reduce to a pair of governing equations, which can be written as

$$\frac{\partial v}{\partial t}+\rho = \lambda\frac{\partial^2 v}{\partial x^2}, \tag{3.14}$$

$$\frac{\partial \rho}{\partial t}-v = \mu\frac{\partial^2 \rho}{\partial x^2}. \tag{3.15}$$

It is clear from these equations and conditions that v and ρ are odd functions of x, so the conditions (3.11) reduce to the following:

$$\frac{\partial \rho}{\partial x}(1,t) = 0, \quad v(1,t) = 0, \tag{3.16}$$

$$\rho(0,t) = 0, \quad v(0,t) = 0. \tag{3.17}$$

If we set λ and μ equal to zero, we obtain the equations for the sloshing mode induced by the initial tilt of the density surfaces. In the absence of diffusion, this motion persists indefinitely, and it is given by

$$\rho = x\cos t, \quad v = -x\sin t. \tag{3.18}$$

We now consider the effect of the two diffusive processes on this motion, concentrating first on the effect of viscosity.

The viscous boundary layers

If we ignore for the moment the effect of density diffusion, (3.14) and (3.15) must be modified by setting $\mu = 0$ and dropping the boundary condition on $\partial\rho/\partial x$. Also, since we are assuming that λ is small, viscous effects will be confined initially to the vicinity of the wall. If we define a boundary-layer variable ξ by

$$x = 1-\lambda^{\frac{1}{2}}\xi, \tag{3.19}$$

the leading-order equations in the boundary layer are

$$v = \frac{\partial \rho}{\partial t}, \tag{3.20}$$

$$\frac{\partial^2 \rho}{\partial t^2}+\rho-\frac{\partial^3 \rho}{\partial t\,\partial \xi^2} = 0, \tag{3.21}$$

and the boundary and initial conditions are

$$\rho = 1 \quad \text{at } \xi = 0, \tag{3.22}$$

$$\rho \sim (1-\lambda^{\frac{1}{2}}\xi)\cos t \quad \text{as } \xi\to\infty, \tag{3.23}$$

$$\rho = 1-\lambda^{\frac{1}{2}}\xi \quad \text{at } t = 0. \tag{3.24}$$

The condition fixing the value of the density at the wall follows from (3.12), (3.16) and (3.20).

The solution for small values of t can be found in terms of a similarity variable

$$\eta = \xi t^{-\frac{1}{2}} \tag{3.25}$$

and an expansion of the form

$$\rho = \bar{g}(\eta)+t^2\bar{g}_1(\eta)+\dots, \tag{3.26}$$

but it is of little interest. More significant is the solution for values of $t \gg 1$. It is convenient to write the density as the sum of two parts, ρ_1 and ρ_2, both of which satisfy (3.21), but which separately satisfy only one of the non-homogeneous conditions (3.22) and (3.23). Thus

$$\left.\begin{aligned} \rho_1 &= 0 \quad \text{at } \xi = 0, \\ \rho_1 &\sim (1-\lambda^{\frac{1}{2}}\xi)\cos t \quad \text{as } \xi \to \infty, \end{aligned}\right\} \tag{3.27}$$

and

$$\left.\begin{aligned} \rho_2 &= 1 \quad \text{at } \xi = 0, \\ \rho_2 &\to 0 \quad \text{as } \xi \to \infty. \end{aligned}\right\} \tag{3.28}$$

The solution for ρ_1 for large t can be found in terms of the similarity variable η already defined and an expansion of the form

$$\rho_1 = \cos t\, g(\eta) + \frac{\sin t}{t} g_1(\eta) + \ldots, \tag{3.29}$$

The leading-order balance of terms in (3.21) is between the first and third terms. The equations for g and g_1 are

$$g'' + \eta g' = 0, \tag{3.30}$$

$$g_1'' + \eta g_1' + 2g_1 = \tfrac{1}{4}\eta^3 g' - \tfrac{3}{4}\eta g, \tag{3.31}$$

and the appropriate boundary conditions are

$$g(0) = g_1(0) = 0, \quad g(\infty) = 1, \quad g_1(\infty) = 0. \tag{3.32}$$

The solutions of these equations are given by

$$g = \operatorname{erf}(\eta/2^{\frac{1}{2}}), \tag{3.33}$$

$$g_1 = (2\pi)^{-\frac{1}{2}}(-\tfrac{1}{8}\eta^3 + c\eta)\exp(-\tfrac{1}{2}\eta^2), \tag{3.34}$$

where c is an arbitrary constant signifying the presence of an eigensolution at this point in the expansion. This solution, and that valid for small t, represent the familiar diffusion from the boundary, the thickness of the layer being proportional to $(\lambda t)^{\frac{1}{2}}$. It becomes comparable to the width of the container, and the boundary-layer assumption ceases to hold, when t is of order λ^{-1}.

For the second part of the solution, we can express ρ_2 in terms of a new similarity variable ζ defined by

$$\zeta = \xi t^{\frac{1}{2}}, \tag{3.35}$$

and the expansion of ρ_2 for large t has the form

$$\rho_2 = f(\zeta) + t^{-2} f_1(\zeta) + \ldots. \tag{3.36}$$

The equations for the leading terms are

$$\tfrac{1}{2}\zeta f''' + f'' - f = 0, \tag{3.37}$$

$$\tfrac{1}{2}\zeta f_1''' - f_1'' - f_1 = \tfrac{1}{4}(\zeta^2 f'' - \zeta f'), \tag{3.38}$$

with boundary conditions given by

$$f(0) = 1, \quad f_1(0) = f(\infty) = f_1(\infty) = 0. \tag{3.39}$$

The solution of (3.37) can be written as a Laplace integral in the form

$$f = A\int_C z^{-1}\exp(z\zeta + z^{-2})\,\mathrm{d}z, \tag{3.40}$$

where A is a constant and the contour C is such that

$$\tfrac{1}{2}z^2\exp(z\zeta+z^{-2}) = 0 \tag{3.41}$$

at both ends of C. There is an essential singularity at $z = 0$, and an acceptable solution that is bounded at infinity is given by choosing C to lie along the imaginary axis from $-\mathrm{i}S$ to $+\mathrm{i}S$, with S positive, and closed in the half-plane $\operatorname{Re} z < 0$. To determine the constant A, we apply the condition $f(0) = 1$. With $\zeta > 0$ and S large, Jordan's lemma shows that the integral round the curved part of C tends to zero as S tends to infinity, so that we require that

$$\lim_{\zeta\to 0} A\int_{-\infty}^{\infty} y^{-1}\exp(\mathrm{i}\zeta y - y^{-2})\,\mathrm{d}y = 1, \tag{3.42}$$

and hence
$$f = -\frac{\mathrm{i}}{\pi}\int_C z^{-1}\exp(z\zeta+z^{-2})\,\mathrm{d}z. \tag{3.43}$$

The value of $f'(0)$ is given by

$$f'(0) = -\frac{\mathrm{i}}{\pi}\int_C \exp(z^{-2})\,\mathrm{d}z; \tag{3.44}$$

by replacing z by $1/z$, so inverting the contour, and after some manipulation, the integral can be written as

$$f'(0) = \frac{2}{\pi}\int_{S^{-1}}^{\infty}\frac{\exp(-y^2)-1}{y^2}\,\mathrm{d}y + \frac{S}{\pi}\int_{\frac{3}{2}\pi}^{\frac{1}{2}\pi}\exp(-\mathrm{i}\theta)\{1-\exp(S^2\,\mathrm{e}^{2\mathrm{i}\theta})\}\,\mathrm{d}\theta. \tag{3.45}$$

If we now let S tend to infinity we obtain

$$f'(0) = \frac{2}{\pi}\int_0^{\infty}\frac{\exp(-y^2)-1}{y^2}\,\mathrm{d}y = -2\pi^{-\frac{1}{2}}. \tag{3.46}$$

To evaluate $f(\zeta)$ for large ζ, we first find the points of stationary phase of the integrand in (3.43), which are at

$$z = \left(\frac{2}{\zeta}\right)^{\frac{1}{3}},\quad \left(\frac{2}{\zeta}\right)^{\frac{1}{3}}\exp\left(\pm\frac{2\mathrm{i}\pi}{3}\right). \tag{3.47}$$

We therefore deform the contour C to pass through the two complex roots, and take the path of steepest descent. The resulting expression for $f(\zeta)$ is

$$f(\zeta) \sim 2^{\frac{4}{3}}3^{-\frac{1}{2}}\pi^{-\frac{1}{2}}\zeta^{-\frac{1}{3}}\exp\{-\tfrac{3}{2}(\tfrac{1}{2}\zeta)^{\frac{2}{3}}\}\cos\{\tfrac{1}{2}3^{\frac{3}{2}}(\tfrac{1}{2}\zeta)^{\frac{2}{3}}-\tfrac{1}{3}\pi\}. \tag{3.48}$$

Hence f has an oscillatory and exponential approach to zero as ζ tends to infinity.

This inner part of the boundary layer decreases in thickness as t increases, and when the first part has expanded to a width comparable to the width of the container, that is, when t is of order λ^{-1}, this second part has thinned to become of order λ^{-1} in thickness. The pinning of the value of the density at the wall to its initial value leads to an ever-increasing gradient of density at the wall, as viscosity acts to damp out all motions in the interior. In this second part of the boundary layer the leading balance of terms in (3.21) is between the second and third terms.

Combining these two parts of the viscous boundary layer, we find that the total boundary value of the density gradient is given by

$$\left.\frac{\partial \rho}{\partial x}\right|_{x=1} = -\frac{\lambda^{-\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}}\frac{\cos t}{t^{\frac{1}{2}}} + \lambda^{-\frac{1}{2}}\frac{2}{\pi^{\frac{1}{2}}}t^{\frac{1}{2}}, \tag{3.49}$$

to leading order and for large t. The large values reached by the second term in (3.49) as t increases indicate that it is not admissible to neglect the diffusion of density. For small t this second diffusive process is only effective within a thin boundary layer, with thickness proportional to $\mu^{\frac{1}{2}}$ when t is of order one. As t increases, this layer thickens and will eventually have the same thickness as the inner part of the viscous boundary layer; the structure of the solution will then change. We now consider first the density boundary layer before these two layers merge and secondly the solution after they have merged.

The density boundary layer

Equations (3.14) and (3.15) show that a boundary layer at the wall can exist, in which the density changes rapidly. The thickness of this layer is proportional to $\mu^{\frac{1}{2}}$, and, for values of $t \gg 1$, the appropriate outer condition for this layer is provided by (3.49). In this layer the density gradient must be reduced from the value given by (3.49) to zero at the wall. It would, of course, be possible to consider the solution for values of t up to order one, but little that is not obvious would be learnt from doing so. For the values of t of interest, it is only the second term on the right-hand side of (3.49) that is significant to leading order, and if we define a boundary-layer variable χ by

$$x = 1-\mu^{\frac{1}{2}}\chi, \tag{3.50}$$

the leading-order equation derived from (3.14) and (3.15) has the form

$$\frac{\partial^3 \rho}{\partial t\,\partial \chi^2} = \frac{\partial^4 \rho}{\partial \chi^4}. \tag{3.51}$$

The conditions applying to the solution of this equation are

$$v = \frac{\partial \rho}{\partial \chi} = 0 \quad \text{at } \chi = 0, \tag{3.52}$$

$$\frac{\partial \rho}{\partial \chi} \to -\frac{2}{\pi^{\frac{1}{2}}}\left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}} t^{\frac{1}{2}} \quad \text{as } \chi \to \infty, \tag{3.53}$$

where the appropriate part of (3.49) has been written in terms of the scaled variable. If we now define a similarity variable σ by

$$\sigma = \chi t^{-\frac{1}{2}}, \tag{3.54}$$

the solution can be written in the form

$$\rho = (\mu/\lambda)^{\frac{1}{2}}\, t\, h(\sigma), \tag{3.55}$$

where

$$h^{\text{iv}} + \tfrac{1}{2}\sigma h''' = 0, \tag{3.56}$$

$$h(0) = -1, \quad h'(0) = 0, \quad h'(\infty) = -2\pi^{-\frac{1}{2}}. \tag{3.57}$$

The solution is easily found to be

$$h = -2\pi^{-\frac{1}{2}}\sigma + \pi^{-\frac{1}{2}}\sigma \exp\left(-\tfrac{1}{4}\sigma^2\right) - \left(1+\tfrac{1}{2}\sigma^2\right)\operatorname{erfc}\left(\tfrac{1}{2}\sigma\right), \tag{3.58}$$

and the value of the density at the wall is then given by

$$\rho = 1-(\mu/\lambda)^{\frac{1}{2}}t. \tag{3.59}$$

This solution is valid so long as the density boundary layer is thinner than the inner part of the viscous boundary layer. Since the thicknesses of the two layers are proportional to $(\mu t)^{\frac{1}{2}}$ and $(\lambda/t)^{\frac{1}{2}}$ respectively, it follows that this solution is valid provided $t \ll (\lambda/\mu)^{\frac{1}{2}}$. When t is of order $(\lambda/\mu)^{\frac{1}{2}}$ the two layers have a thickness of order $(\lambda\mu)^{\frac{1}{4}}$. This suggests that, to examine the solution when the two layers merge, we should introduce new variables defined by

$$x = 1-(\lambda\mu)^{\frac{1}{4}}q, \quad t = \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}}\tau, \tag{3.60}$$

with the scaled variables q and τ both of order one.

Equations (3.14) and (3.15) can be combined to give the single equation

$$\frac{\partial^2\rho}{\partial t^2}+\rho-(\lambda+\mu)\frac{\partial^3\rho}{\partial t\,\partial x^2}+\lambda\mu\frac{\partial^4\rho}{\partial x^4} = 0, \tag{3.61}$$

which, with the above scalings introduced, becomes

$$\rho-\frac{\partial^3\rho}{\partial\tau\,\partial q^2}+\frac{\partial^4\rho}{\partial q^4} = 0, \tag{3.62}$$

to leading order. The boundary conditions at $q = 0$ are, in terms of the scaled variables,

$$\frac{\partial\rho}{\partial q} = 0, \quad \frac{\partial\rho}{\partial\tau}-\frac{\partial^2\rho}{\partial q^2} = 0, \tag{3.63}$$

and, when $\tau = 0$, $\rho = 1$ at $q = 0$. A solution by means of a Laplace transform is easily found, and has the form

$$\rho = \frac{1}{2\pi \mathrm{i}}\int\frac{\lambda_2\exp(-\lambda_1 q)-\lambda_1\exp(-\lambda_2 q)}{(\lambda_2-\lambda_1)(s+\lambda_1\lambda_2)}\mathrm{e}^{s\tau}\,\mathrm{d}s, \tag{3.64}$$

where $$\lambda_1 = \tfrac{1}{2}(s+2)^{\frac{1}{2}}+\tfrac{1}{2}(s-2)^{\frac{1}{2}}, \quad \lambda_2 = \tfrac{1}{2}(s+2)^{\frac{1}{2}}-\tfrac{1}{2}(s-2)^{\frac{1}{2}}, \tag{3.65}$$

and the integral is taken along the imaginary axis. The quantities of most interest are the values of the density and the velocity gradient at the wall. The former is given by

$$\rho = \frac{1}{2\pi \mathrm{i}}\int\frac{\mathrm{e}^{s\tau}}{s+\lambda_1\lambda_2}\mathrm{d}s = \mathrm{e}^{-\tau}, \tag{3.66}$$

which agrees with (3.59) for small τ. The velocity gradient at the wall is given by

$$\delta = \left.\frac{\partial v}{\partial x}\right|_{x=1} = \left(\frac{\mu}{\lambda^3}\right)^{\frac{1}{4}}\left.\frac{\partial^3\rho}{\partial q^3}\right|_{q=0}, \tag{3.67}$$

where use has been made of (3.15), and, from (3.64), we then deduce that

$$\delta = \left(\frac{\mu}{\lambda^3}\right)^{\frac{1}{4}}\frac{1}{2\pi \mathrm{i}}\int\frac{(s+2)^{\frac{1}{2}}}{s+1}\mathrm{e}^{s\tau}\,\mathrm{d}s. \tag{3.68}$$

Inversion of the transform finally gives the result that

$$\delta = \left(\frac{\mu}{\lambda^3}\right)^{\frac{1}{4}}\{\mathrm{e}^{-\tau}\operatorname{erf}\tau^{\frac{1}{2}}+(\pi\tau)^{-\frac{1}{2}}\mathrm{e}^{-2\tau}\}. \tag{3.69}$$

The main features of the boundary layers have now been obtained, and we can proceed to a discussion of the stages by which the fluid attains equilibrium.

4. Timescales for the attainment of equilibrium

The initial tilt of the surfaces of constant density induces a sloshing motion with a period proportional to $(\beta' g/a)^{-\frac{1}{2}}$, and this quantity has been chosen as the reference timescale. For values of t of order one on this scale, the interior motion is bordered by thin layers at the walls. There is a viscous layer of width $\lambda^{\frac{1}{2}}$ in which the interior velocity is reduced to zero at the wall, and a layer of width $\mu^{\frac{1}{2}}$ in which density diffusion is important and in which the density gradient at the wall is made zero. The density layer is much thinner than the viscous layer. As t increases, the viscous layer splits into two parts, one of which thickens with time, so that its width is proportional to $(\lambda t)^{\frac{1}{2}}$, while the other thins, with its width being proportional to $(\lambda/t)^{\frac{1}{2}}$. The outer part is where the viscous damping of the interior motion is produced, and the inner part is characterized by the appearance of large density gradients near the wall. Also, as t increases to values large compared with one, the density layer grows, its width being proportional to $(\mu t)^{\frac{1}{2}}$. The description of the subsequent structure of the solution depends on whether the inner viscous layer and the density layer merge before or after the outer part of the viscous layer has grown so large that it is comparable in width to the size of the container. Since the merging of the two layers takes place when t is proportional to $(\lambda/\mu)^{\frac{1}{2}}$, while the outer layer becomes as thick as the container when t is of order λ^{-1}, the next stage in the development of the solution depends on whether λ^3 is less than or greater than μ. If $\lambda^3 \ll \mu$ the density at the wall, given by (3.66), is exponentially small in a time of order $(\lambda/\mu)^{\frac{1}{2}}$, but the viscous boundary layer then only occupies a small fraction of the container. Thus the large density gradients at the wall have disappeared, and the density there is reduced to zero, but there is still a considerable sloshing motion in the fluid, with a corresponding movement of the surfaces of constant density. It is only when the later time of order λ^{-1} has been reached that the damping of this motion is completed. If $\lambda^3 \gg \mu$ the reverse situation applies. When t is of order λ^{-1} the damping of the interior motion takes place, but the boundary value of the density has then been only slightly reduced from its initial value, and there are large density gradients near the wall. It is only when the later time of order $(\lambda/\mu)^{\frac{1}{2}}$ has been reached that the final equilibrium state is attained. The widths of the various layers, and their dependence on t are shown schematically in figure 2.

Since the analysis of the motion has been carried out in boundary-layer terms, it has not been possible to give the solution when these layers are no longer thin. An alternative method of analysis would have been to use a Laplace transform of the complete equations and deduce from the resulting integral the existence of boundary layers. It would then have been possible to obtain the details of the final damping of the interior motion as well. But this does not add a great deal to the understanding of the way in which equilibrium is attained, and the boundary-layer approach is sufficient for the determination of the important processes involved. It is also easier to determine the boundary-layer scalings directly from the equations than by examination of the Laplace transform.

5. Conclusions and comments

We have seen how a contained stratified fluid may contain an initial distribution of density in which the density surfaces are tilted from the horizontal. For the special case of a long container with vertical walls, we have examined the way by which this initial state, and the fluid motion induced by it, are acted on by diffusive processes until an equilibrium state has been reached. The timescales for the attainment of

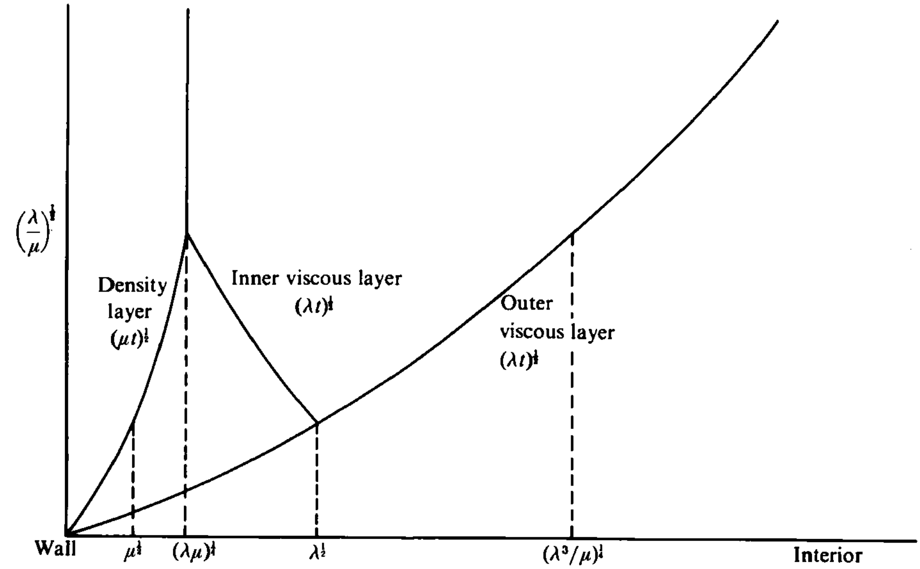

FIGURE 2. Sketch showing schematically the widths of the boundary layers near the wall when $\lambda^3 \ll \mu$. For $\lambda^3 \gg \mu$ the outer viscous layer has filled the interior (and lost its boundary-layer character) before the inner viscous layer and the density layer have merged, at time $t = (\lambda/\mu)^{\frac{1}{2}}$.

equilibrium depend on both diffusivities, or rather on non-dimensional parameters proportional to them. The most important conclusion is that, in a laboratory experiment, it may not be sufficient only to wait until any transient motion in the fluid has disappeared before commencing the experiment. Even when all motion has ceased, there may still be regions near the container walls where the density is far from its equilibrium value. This situation occurs when $\lambda^3 \gg \mu$. Taking the values of λ and μ for salt water, for which $\lambda/\mu = 560$, this condition requires that $\lambda \gg \frac{1}{24}$.

Although we have dealt here only with a very simple initial state, the same processes would be active for an initial state in which the density surfaces were not plane and the fluid not at rest. The container has been chosen to have a simple shape, and other effects would be important for other containers. If the walls of the container are not vertical the effect of gravity in producing horizontal density surfaces and the effect of diffusion, which requires zero normal density gradients at the walls, are in conflict and a flux of fluid parallel to the wall will be produced. A mixing of the basic vertical density distribution assumed here would thus be produced, whereas with vertical walls it is only the end effects that can eventually cause this mixing. The absence of these boundary currents in the problem discussed here means that there is no interaction between the central portion of the container and its ends. The adjustment of the fluid to its equilibrium state near the ends probably involves the same processes that are important in the rest of the container, but the complex geometry makes it more difficult to analyse. There will, of course, be a gradual diffusion from the ends, where the vanishing of the density gradients will eventually affect the density distribution everywhere, but this is a very slow process. After a time of order μ^{-1}, which is much longer than any of the other timescales we have identified, the effect of the horizontal boundaries will only have spread into the container a distance of the same order as the container width, and so is negligible for a container much longer than it is wide.

The vertical density gradient has been assumed to be uniform. Any variation would produce a horizontal motion of the fluid, since the vertical velocity would then be height-dependent. The consequent advection of density would radically alter the structure of the flow, and the assumed uniformity is essential for the analysis presented here.

There is often an analogy between stratified and rotating fluids. The corresponding problem to that examined here would be the establishment of a rigid rotation from an initial state in which there was a non-uniformity in the angular velocity. The adjustment would be produced by Ekman layers at the walls. However, the analogy between the solutions of the two problems does not exist, since we are dealing with unsteady boundary layers. The diffusion of radial velocity and of angular velocity are of equal strengths, whereas the diffusion of density and of velocity in the problem considered here are of very different strengths. As Veronis (1970) points out, this is an example where the analogy does not apply.

Finally, it should be pointed out that no attempt has been made to discuss the stability of the damped sloshing motion generated by the tilting of the density surfaces. The discussion of this motion can be regarded as confirming the stability of the equilibrium state of which it is a perturbation. But whether or not this motion can itself provide the energy source to fuel any other type of perturbed motion is an open question.

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